Generalizations of classical integrable nonholonomic rigid body systems

1998 J. Phys. A: Math. Gen. 31 9861

(http://iopscience.iop.org/0305-4470/31/49/010)

J. Phys. A: Math. Gen. **31** (1998) 9861–9869. Printed in the UK PII: S0305-4470(98)93777-5

Generalizations of classical integrable nonholonomic rigid body systems

Vladimir Dragović†, Borislav Gajić‡ and Božidar Jovanović§
Mathematical institute SANU, Kneza Mihaila 35, 11000 Beograd, Yugoslavia

Received 28 April 1998, in final form 14 September 1998

Abstract. Nonholonomic systems with an invariant measure: the Suslov, Chaplygin and Veselov–Veselova problem are considered. New families of integrable potential perturbations of these non-Hamiltonian systems are constructed. We also obtain a similar result for a classical Hamiltonian system of the motion of a rigid body fixed at a point.

1. Introduction

One of the common classifications of mechanical systems to holonomic (with integrable constraints) and nonholonomic (with nonintegrable constraints) was provided by Herz. He noticed important differences between these situations: equations of motion in nonholonomic cases obtained from the Lagrange–D'Alembert principle were not equivalent to those derived from the variational principle, and nonholonomic systems are not Hamiltonian. As is well known, a Hamiltonian system preserves the standard measure. If, in ($m = 2n$)-dimensional phase space, we have $n = m/2$ functionally independent integrals in involution, then the Hamiltonian system is completely integrable. In general, a nonholonomic system with k nonholonomic constraints does not have an invariant measure in ($m - k$)-dimensional phase space, and one needs $m - k - 1$ functionally independent integrals for complete integrability.

In this paper, we consider nonholonomic systems with an invariant measure and, according to the Jacobi theorem for the integrability by quadratures, we need only $m-k-2$ independent integrals [1]. We have constructed families of integrable perturbations for a few known integrable nonholonomic problems: the Suslov problem [2], the Chaplygin problem [3, 4] and the Veselov and Veselova case [5]. The famous integrable perturbations found by Kharlamova-Zabelina, Kozlov and Veselov–Veselova are special cases of our solutions. Also, we have obtained new integrable perturbations of a rigid body motion around a fixed point, even without a nonholonomic constraint.

The method we use here is a modification of that which we have used for perturbating the Jacobi problem for geodesics on the ellipsoid [6], and billiard systems [7–9]. The basic idea (due to Kozlov [10]) is the following. Suppose that we have an integrable natural mechanical system with integrals $F_i(\dot{x}, x)$, $i = 1, \dots, k$. Is it possible to add a potential $V = V(x)$ such that the new system has integrals $\tilde{F}_i(\dot{x}, x) = F_i + U_i(x)$, where $U_i(x)$ are

† E-mail address: vladad@mi.sanu.ac.yu
‡ E-mail address: gajab@mi.sanu.ac.yu
§ E-mail address: bozaj@mi.sanu.ac.yu

0305-4470/98/499861+09$19.50

functions depending only on x? The answer gives a system of partial differential equations (PDEs) on V. We are looking for solutions in the form of Laurent polynomials.

Let us note that some other generalizations of the considered classical nonholonomic rigid body systems could be found in [5, 11, 12].

2. The Suslov problem

The Suslov problem describes the rotation of a rigid body fixed at a point subjected to the constraint $\langle n, \omega\rangle = 0$, where ω is angular velocity and n is a constant vector in a moving frame [2]. From the Frobenius theorem it follows that

$$\mathcal{D} = \{\langle n, \omega\rangle = f(\dot\phi, \dot\psi, \dot\theta, \phi, \psi, \theta) = 0\} \subset TSO(3)\{\dot\phi, \dot\psi, \dot\theta, \phi, \psi, \theta\}$$

is the nonintegrable distribution in the tangent bundle of the rigid body configuration space $SO(3)$ (ϕ, ψ, θ are Euler's angles).

We will treat the rigid body dynamics in the first-order Euler–Poisson equations rather than in the second-order Euler–Lagrange equations on the $SO(3)$ (derivation of the equations can be found, for example, in [1, 2]).

Let $\{\alpha, \beta, \gamma\}$ be the base of the fixed reference frame. Let a rigid body be fixed at the point $O = (0, 0, 0)$, and placed in the potential force field with the potential $v = v(x_1, x_2, x_3)$. The total potential energy of the rigid body is

$$V = \int_B v(\langle r, \alpha\rangle, \langle r, \beta\rangle, \langle r, \gamma\rangle)\, \mathrm{d}m(r) = V(\alpha, \beta, \gamma)$$

where r is the radius vector of the point of the rigid body according to the fixed point. We shall consider potentials which depend only on γ: $V = V(\gamma)$ (for example, in the case of a gravity field $v = gx_3$, we have $V(\gamma) = \text{constant}\ \langle r_C, \gamma\rangle$, where r_C is the radius vector of the mass centre).

The equations of the motion in the moving frame are [1, 13]

$$I\dot\omega = I\omega \times \omega + \gamma \times \frac{\partial V}{\partial \gamma} + \lambda n \qquad \dot\gamma = \gamma \times \omega, \ \ \langle n, \omega\rangle = 0 \tag{1}$$

where I is the inertia tensor ($I = I^t$). The Lagrange multiplier λ is determined from the constraint

$$\lambda = \frac{1}{\langle I^{-1}n, n\rangle} \left\langle I^{-1}n, \omega \times I\omega + \frac{\partial V}{\partial \gamma} \times \gamma \right\rangle.$$

There are always two independent integrals of equations (1):

$$F_1 = \frac{\langle I\omega, \omega\rangle}{2} + V(\gamma) \qquad F_2 = \langle \gamma, \gamma\rangle.$$

We will take $F_2 = \langle \gamma, \gamma\rangle = 1$ (according to the description of the physical model we have given). Thus, the phase space is

$$\mathcal{M} = \{(\omega, \gamma) \in R^3\{\omega\} \times R^3\{\gamma\} | \langle n, \omega\rangle = 0, \langle \gamma, \gamma\rangle = 1\}. \tag{2}$$

If $V = 0$, then equations (1) form a closed system in $R^3\{\omega\}$, and for general n they do not have an invariant measure. This problem was solved by Suslov.

The system (1) preserves the standard measure in $\mathcal{M}$ if n is an eigenvector of the inertia operator. Thus, in what follows we will assume that n is an eigenvector of the operator I. Then, for the integrability, we need one more integral. We can choose the base $\{e_1, e_2, e_3\}$ of the moving frame such that $I = \mathrm{diag}(I_1, I_2, I_3)$ and $n = e_3$.

The known integrable cases are as follows.

(i) The Kharlamova-Zabelina case [1, 14] where $V(\gamma) = \langle b, \gamma\rangle$, with b such that $\langle n, b\rangle = 0$. Then $F_3 = \langle I\omega, b\rangle$.

(ii) The Lagrange case (noted by Kozlov [13]) when $I_1 = I_2$, potential is $V = \langle b, \gamma\rangle$, where $b = \epsilon n$. The integral is $F_3 = \langle I\omega, \gamma\rangle$.

(iii) The Klebsh–Tisserand–Kozlov case [1, 13] with $V(\gamma) = \frac{\epsilon}{2}\langle I\gamma, \gamma\rangle$ and $F_3 = \frac{1}{2}\langle I\omega, I\omega\rangle - \frac{1}{2}\langle A\gamma, \gamma\rangle$, where $A = \epsilon I^{-1}\det I$.

Remark 1. Let us note that in case (ii) the Lagrange multiplier is $\lambda = 0$, so it is actually a holonomic system. It remains integrable if the constraint is $\langle n, \omega\rangle = c =$ constant.

Remark 2. The physical meaning of $V(\gamma) = \frac{\epsilon}{2}\langle I\gamma, \gamma\rangle$ is that it represents the potential of a rigid body in a central Newtonian force field to within $O(r^4/R^4)$, where r is the typical body dimension and R is a distance from the body to the centre of the attraction [1].

We are looking for potentials $V(\gamma)$ for which there exists an integral of the system (1) of the form

$$\tilde{F}_3 = \tfrac{1}{2}\langle I\omega, I\omega\rangle + F(\gamma)$$

where F depends only on γ.

If $I_1 = I_2$ the integral $\tilde{F}_3$ and the energy integral are dependent, so in this way we cannot get new integrable cases.

From $\dot{\tilde{F}}_3 = 0$ we have

$$\begin{aligned}&\omega_1\left(I_1\gamma_2\frac{\partial V}{\partial\gamma_3} - I_1\gamma_3\frac{\partial V}{\partial\gamma_2} + \gamma_3\frac{\partial F}{\partial\gamma_2} - \gamma_2\frac{\partial F}{\partial\gamma_3}\right)\\ &-\omega_2\left(I_2\gamma_1\frac{\partial V}{\partial\gamma_3} - I_2\gamma_3\frac{\partial V}{\partial\gamma_1} + \gamma_3\frac{\partial F}{\partial\gamma_1} - \gamma_1\frac{\partial F}{\partial\gamma_3}\right) = 0.\end{aligned}$$

It gives us two equations:

$$\begin{aligned}I_1\left(\gamma_2\frac{\partial V}{\partial\gamma_3} - \gamma_3\frac{\partial V}{\partial\gamma_2}\right) &= \gamma_2\frac{\partial F}{\partial\gamma_3} - \gamma_3\frac{\partial F}{\partial\gamma_2}\\ I_2\left(\gamma_1\frac{\partial V}{\partial\gamma_3} - \gamma_3\frac{\partial V}{\partial\gamma_1}\right) &= \gamma_1\frac{\partial F}{\partial\gamma_3} - \gamma_3\frac{\partial F}{\partial\gamma_1}.\end{aligned} \tag{3}$$

Introducing $c = (F - I_1V)/(I_2 - I_1)$, $d = (F - I_2V)/(I_2 - I_1)$ equations (3) reduce to

$$\frac{\partial c}{\partial\gamma_2}\gamma_3 - \frac{\partial c}{\partial\gamma_3}\gamma_2 = 0 \qquad \frac{\partial d}{\partial\gamma_1}\gamma_3 - \frac{\partial d}{\partial\gamma_3}\gamma_1 = 0. \tag{4}$$

The general solution of the system (4) is $c = c(\gamma_1, \gamma_2^2 + \gamma_3^2)$, $d = d(\gamma_2, \gamma_1^2 + \gamma_3^2)$. Thus we have the following theorem.

Theorem 1. If $In = I_3n$ and $I_1 \neq I_2$ then equations (1) of the Suslov problem are integrable for potentials:

$$V(\gamma) = c(\gamma_1, \gamma_2^2 + \gamma_3^2) - d(\gamma_2, \gamma_1^2 + \gamma_3^2)$$

where c, d are arbitrary functions of two variables. The corresponding third integral is

$$\tilde{F}_3(\omega, \gamma) = \tfrac{1}{2}\langle I\omega, I\omega\rangle + I_2c(\gamma_1, \gamma_2^2 + \gamma_3^2) - I_1d(\gamma_2, \gamma_1^2 + \gamma_3^2).$$

The polynomial solutions of the system (3) are described in the following.

Corollary 1. For $I_1 \neq I_2$, the base of the polynomial solutions of the linear PDE system (3) is given by the following homogeneous polynomials:

$$V_{2L}(\gamma) = \sum_{\substack{m+n+k=L \\ m,n,k\geqslant 0}} \left(\binom{n+k}{k} c_{n+k} - \binom{m+k}{k} d_{m+k} \right) \gamma_1^{2m}\gamma_2^{2n}\gamma_3^{2k}$$
$$V_{2L+1}(\gamma) = \sum_{\substack{m+n+k=L \\ m,n,k\geqslant 0}} \left(\gamma_1 \binom{n+k}{k} c_{n+k} - \gamma_2 \binom{m+k}{k} d_{m+k} \right) \gamma_1^{2m}\gamma_2^{2n}\gamma_3^{2k} \tag{5}$$

where $c_0, \ldots, c_L, d_0, \ldots, d_L$ are arbitrary constants, and the corresponding Fs are

$$F_{2L}(\gamma) = \sum_{\substack{m+n+k=L \\ m,n,k\geqslant 0}} \left(I_2 \binom{n+k}{k} c_{n+k} - I_1 \binom{m+k}{k} d_{m+k} \right) \gamma_1^{2m}\gamma_2^{2n}\gamma_3^{2k}$$
$$F_{2L+1}(\gamma) = \sum_{\substack{m+n+k=L \\ m,n,k\geqslant 0}} \left(I_2\gamma_1 \binom{n+k}{k} c_{n+k} - I_1\gamma_2 \binom{m+k}{k} d_{m+k} \right) \gamma_1^{2m}\gamma_2^{2n}\gamma_3^{2k}. \tag{6}$$

Examples.

(i) For $N = 1$, we have the Kharlamova-Zabelina case: $V_1(\gamma) = b_1\gamma_1 + b_2\gamma_2 = \langle b, \gamma \rangle$, and $\langle n, b \rangle = 0$.

(ii) For $N = 2$, the potential is $V(\gamma) = \frac{1}{2}(a_1\gamma_1^2 + a_2\gamma_2^2 + a_3\gamma_3^2)$, and

$$\tilde{F}_3(\omega, \gamma) = \tfrac{1}{2}\langle I\omega, I\omega \rangle - \tfrac{1}{2}(I_2 a_3 \gamma_1^2 + (I_2 a_1 - I_1 a_2 + I_1 a_3)\gamma_2^2 + I_2 a_1 \gamma_3^2).$$

By choosing $a_1 = \epsilon I_1$, $a_2 = \epsilon I_2$, $a_3 = \epsilon I_3$, we get the Klebsh–Tisserand–Kozlov case.

(iii) Solutions of Laurent type are, for example, $V(\gamma) = c\gamma_1^m - d\gamma_2^n$, $m, n < 0$.

3. The Suslov problem with gyroscopic force

There is one new integrable case of the Suslov problem. Including the gyroscopic force, with the momentum $\epsilon\gamma \times \omega$, equations (1) become

$$I\dot{\omega} = I\omega \times \omega + \gamma \times \frac{\partial V}{\partial \gamma} + \epsilon\gamma \times \omega + \lambda n \qquad \dot{\gamma} = \gamma \times \omega, \;\; \langle n, \omega \rangle = 0$$
$$\lambda = \frac{1}{\langle I^{-1}n, n \rangle} \left\langle I^{-1}n, \omega \times (I\omega + \epsilon\gamma) + \frac{\partial V}{\partial \gamma} \times \gamma \right\rangle. \tag{7}$$

They describe, for example, the motion of a rigid body with the magnetic momentum $-\epsilon\omega$ in an additional homogeneous magnetic field in the direction of γ. Since a gyroscopic force is conservative, F_1 remains the first integral.

Theorem 2. If $In = I_k n$ then the Suslov problem with the gyroscopic force (7) is integrable for potentials $V = 0$, $V(\gamma) = \langle b, \gamma \rangle$, where $\langle n, b \rangle = 0$ and $V(\gamma) = \frac{1}{2}(a_1\gamma_1^2 + a_2\gamma_2^2 + a_3\gamma_3^2)$. Especially, for $V = 0$, all trajectories are closed.

Proof. As for (1), it could be proved that if n is an eigenvector of the inertia tensor I, then equations (7) preserve the standard measure in $\mathcal{M}$ (defined by (2)). Let, for example, $In = I_3 n$.

(1) $V = 0$. Then we have two supplementary integrals $F_3 = I_1\omega_1 - \epsilon\gamma_1 = c_1$ and $F_4 = I_2\omega_2 - \epsilon\gamma_2 = c_2$. Let $2F_1 = \langle I\omega, \omega\rangle + 2V(\gamma) = h$. By integration we obtain the period of the closed trajectories

$$T(h, c_1, c_2) = \sqrt{I_1 I_2}\int_0^{2\pi} \frac{\mathrm{d}\varphi}{\sqrt{\epsilon^2 - (c_1 - \sqrt{I_1 h}\cos\varphi)^2 - (c_2 - \sqrt{I_2 h}\sin\varphi)^2}}.$$

(2) $V(\gamma) = \langle b, \gamma\rangle = b_1\gamma_1 + b_2\gamma_2$. The third integral is $F_3 = b_1(I_1\omega_1 - \epsilon\gamma_1) + b_2(I_2\omega_2 - \epsilon\gamma_2)$.

(3) $V(\gamma) = \frac{1}{2}(a_1\gamma_1^2 + a_2\gamma_2^2 + a_3\gamma_3^2)$. The additional independent integral is

$$\begin{aligned} F_3 = {} & (I_1^2(a_3 - a_1) + I_1\epsilon^2)\omega_1^2 + (I_2^2(a_3 - a_2) + I_2\epsilon^2)\omega_2^2 - 2\epsilon I_1(a_3 - a_1)\omega_1\gamma_1 \\ & -2\epsilon I_2(a_3 - a_2)\omega_2\gamma_2 - (a_3 - a_1)(a_3 - a_2)(I_2\gamma_1^2 + I_1\gamma_2^2). \end{aligned}$$

□

4. Perturbations of the Chaplygin problem

We are interested in integrable potential perturbations of the Chaplygin problem of a balanced, dynamically asymmetric ball ($I_1 \neq I_2 \neq I_3$) rolling on a rough surface. The nonholonomic constraint is given by the condition that the velocity of the point of contact is equal to zero. The equations of the motion in a potential field with potential $V(\gamma)$ are [1, 13]:

$$\dot{k} + \omega \times k = \gamma \times \frac{\partial V}{\partial \gamma} \qquad \dot{\gamma} = \gamma \times \omega \tag{8}$$

where $k = I\omega + ma^2\gamma \times (\omega \times \gamma)$ is the angular momentum of the ball relative to the point of contact, a is the radius, m is the mass and I is the inertia tensor of the ball relative to its centre. Equations (8) have the invariant measure with the density

$$M = \frac{1}{\sqrt{(ma^2)^{-1} - \langle\gamma, (I + ma^2E)^{-1}\gamma\rangle}}$$

where E is the identity matrix. They always possess the following three integrals

$$F_1 = \tfrac{1}{2}\langle k, \omega\rangle + V(\gamma) \qquad F_2 = \langle k, \gamma\rangle \qquad F_3 = \langle\gamma, \gamma\rangle (= 1).$$

Chaplygin considered the motion without the potential force. He found the fourth integral $F_4 = \langle k, k\rangle$, and solved the problem by quadratures [3].

Kozlov generalized the Chaplygin problem by adding the potential $V = \frac{\epsilon}{2}\langle I\gamma, \gamma\rangle$. Then the fourth integral is $F_4 = \langle k, k\rangle - \langle A\gamma, \gamma\rangle$, where A is a diagonal matrix with diagonal elements $A_1 = \epsilon(I_2 + ma^2)(I_3 + ma^2)$, $A_2 = \epsilon(I_1 + ma^2)(I_3 + ma^2)$, $A_3 = \epsilon(I_1 + ma^2)(I_2 + ma^2)$.

We are looking for the fourth integral, using the same method as we did in section 2, in the form

$$\tilde{F}_4 = \tfrac{1}{2}\langle k, k\rangle + F(\gamma).$$

The condition $\dot{\tilde{F}}_4 = 0$ is equivalent to the following system:

$$K_1\left(\frac{\partial V}{\partial\gamma_3}\gamma_2-\frac{\partial V}{\partial\gamma_2}\gamma_3\right)=\frac{\partial F}{\partial\gamma_3}\gamma_2-\frac{\partial F}{\partial\gamma_2}\gamma_3$$
$$K_2\left(\frac{\partial V}{\partial\gamma_1}\gamma_3-\frac{\partial V}{\partial\gamma_3}\gamma_1\right)=\frac{\partial F}{\partial\gamma_1}\gamma_3-\frac{\partial F}{\partial\gamma_3}\gamma_1 \qquad (9)$$
$$K_3\left(\frac{\partial V}{\partial\gamma_2}\gamma_1-\frac{\partial V}{\partial\gamma_1}\gamma_2\right)=\frac{\partial F}{\partial\gamma_2}\gamma_1-\frac{\partial F}{\partial\gamma_1}\gamma_2$$

where $K_i = I_i + ma^2$, $i = 1, 2, 3$. The first two equations in (9) are the same as equations (3) in the Suslov problem. We shall derive polynomial solutions. From corollary 1 we have that any polynomial solution should be of even degree in γ_3. Using the symmetry in γ_1, γ_2 and γ_3 of (9), we also obtain an even degree in γ_1, γ_2.

By substituting V_{2L} from (5) and F_{2L} from (6) to the third equation, we get conditions on $c_0, \ldots, c_L, d_0, \ldots, d_L$:

$$(I_3-I_2)(m+1)\binom{n+k-1}{k}c_{n+k-1}+(I_1-I_3)(m+1)\binom{m+k+1}{k}d_{m+k+1}$$
$$+n(I_3-I_1)\binom{m+k}{k}d_{m+k}+n(I_2-I_3)\binom{n+k}{k}c_{n+k}=0$$
$$1\leqslant n\leqslant L \qquad 0\leqslant m,k\leqslant L \qquad n+m+k=L. \qquad (10)$$

The system (10) consists of $L(L+1)/2$ equations with $2L+2$ unknown variables. Let $\binom{L-1}{-1}=0$.

Lemma 1. The rank of the system (10) is $2L-1$. The general solution depends on three independent parameters A, B, C:

$$c_{L-i}=\binom{L}{i}(1-i)(I_1-I_3)A+\binom{L-1}{i-1}(I_1-I_3)B$$
$$d_i=\binom{L}{i}(I_3-I_2)iA+\binom{L-1}{i-1}(I_2-I_3)B+\binom{L}{i}C. \qquad (11)$$

Proof. There are $2L-1$ independent equations in (10); for example, the equations with indexes $(m,n,k)=(0,L-i,i)$ and $(m,n,k)=(i,L-i,0)$, where $i=0,\ldots,L-1$. The solution of the subsystem is given by (11). One can prove that this is the general solution of (10) by substitution of (11) in to the rest of the equations. □

From lemma 1 and the previous consideration we have the following.

Theorem 3. Equations (8) of the Chaplygin problem, for $I_1 \neq I_2 \neq I_3$, are integrable for potentials $V=\sum_L a_L V_{2L}(\gamma|A_L,B_L,C_L)$, where:

$$V_{2L}=\sum_{\substack{m+n+k=L\\ m,n,k\geqslant 0}}\left(\binom{n+k}{k}c_{n+k}-\binom{m+k}{k}d_{m+k}\right)\gamma_1^{2m}\gamma_2^{2n}\gamma_3^{2k}. \qquad (12)$$

The corresponding fourth integral is $\tilde{F}_4=\frac{1}{2}\langle k,k\rangle+\sum_L a_L F_{2L}(\gamma|A_L,B_L,C_L)$, where:

$$F_{2L}=\sum_{\substack{m+n+k=L\\ m,n,k\geqslant 0}}\left(K_2\binom{n+k}{k}c_{n+k}-K_1\binom{m+k}{k}d_{m+k}\right)\gamma_1^{2m}\gamma_2^{2n}\gamma_3^{2k}.$$

cs and ds depend on A_L, B_L, C_L by (11).

Examples.

(i) For $N = 2$ the solution is the Klebsh potential

$$V_2(\gamma) = a_1\gamma_1^2 + a_2\gamma_2^2 + a_3\gamma_3^2 \qquad a_1(I_2 - I_3) + a_2(I_3 - I_1) + a_3(I_1 - I_2) = 0.$$

(ii) For $N = 4$ the new integrable potential is $V_4(\gamma) = \sum_{i+j+k=2} a_{2i2j2k}\gamma_1^{2i}\gamma_2^{2j}\gamma_3^{2k}$,

$$\begin{aligned}
a_{400} &= c_0 - d_2 = (2I_2 - I_3 - I_1)A + (I_1 - I_2)B - C\\
a_{040} &= c_2 - d_0 = (I_1 - I_3)A - C\\
a_{004} &= c_2 - d_2 = (I_1 + 2I_2 - 3I_3)A + (I_3 - I_2)B - C\\
a_{220} &= c_1 - d_1 = 2(I_2 - I_3)A + (I_1 - I_2)B - 2C\\
a_{202} &= c_1 - 2d_2 = 4(I_2 - I_3)A + (I_1 - 2I_2 + I_3)B - 2C\\
a_{022} &= 2c_2 - d_1 = 2(I_1 + I_2 - 2I_3)A + (I_3 - I_2)B - 2C.
\end{aligned}$$

Note. (Rolling of a dynamically symmetric ball.) The general solution of (9) for $I_1 \neq I_2 = I_3$ is $V = V(\gamma_1, \gamma_2^2 + \gamma_3^2)$, $F = K_3V$. This fact simply implies the integrability of the following two problems. The first one, which is solved by Chaplygin [4], is a symmetric but nonbalanced ball, with the centre of the mass on the axis of dynamical symmetry, rolling on a horizontal plane in a gravitational field (this corresponds to equations (8) for $V(\gamma) = \epsilon\gamma_1$). The second is the motion of that ball under the influence of a potential, which is invariant under rotations about the axis of dynamical symmetry. Note that the constraint is not invariant under the rotations.

In this case, the perturbed system could be easily solved by quadratures. For the fourth integral we can take

$$J_4 = \tilde{F}_4 - K_3F_1 = \tfrac{1}{2}\langle k, k - K_3\omega\rangle$$

for all potentials $V(\gamma_1, \gamma_2^2 + \gamma_3^2)$. Simplifying J_4 we can get the equivalent integral: $\tilde{J}_4 = \omega_1^2 + \frac{ma^2(I_1-I_3)}{K_1I_3}\omega_1^2\gamma_1^2$. Let, for example, $\alpha = I_1 - I_3 > 0$. Let $\beta^2 = ma^2\alpha/K_1I_3$. On the invariant surface

$$\mathcal{T} = \{(\omega, \gamma) \in R^6 | F_1 = h, F_2 = c, F_3 = 1, \tilde{J}_4 = d^2\}$$

we can introduce variables u and v by the formulae

$$\omega_1 = d\cos u \qquad \gamma_1 = \beta^{-1}\tan u \qquad \omega_2 = R(u)\cos v \qquad \omega_3 = R(u)\sin v \tag{13}$$

where $R(u)^2 = \frac{1}{K_3}(2h + ma^2P(u)^2 - 2V(u) - K_1d^2\cos^2 u)$, $P(u) = \frac{c}{I_3} - \frac{\alpha d}{I_3\beta}\sin u$, $V(u) = V(\beta^{-1}\tan u, 1 - \beta^{-2}\tan^2 u)$. From the first equation for γ (8), we obtain

$$I_3^2\dot{u}^2 = I_3^2\cos^4 u R(u)^2(\beta^2 - \tan^2 u) - \cos^4 u(I_1d\sin u - c\beta)^2 \tag{14}$$

Thus, we can get $u = u(t)$. From the second and the third equation for k (8), it follows that

$$\begin{aligned}
\gamma_2 &= K_3(M\dot{\omega}_2 - N\dot{\omega}_3) + \alpha\omega_1(M\omega_3 + N\omega_2)\\
\gamma_3 &= K_3(M\dot{\omega}_3 + N\dot{\omega}_2) + \alpha\omega_1(N\omega_3 - M\omega_2)
\end{aligned} \tag{15}$$

where $M = ma^2\dot{P}/\Delta$, $N = \beta\cos^2 u\partial_u V/\Delta$, $\Delta = m^2a^4\dot{P}^2 + (\beta\cos^2 u\partial_u V)^2$ are known functions of time. By substituting (15) in to $F_2 = c$ we get

$$\dot{v} = \frac{\alpha d}{K_3}\cos u + \frac{M\dot{R}}{NR} + \frac{I_1d}{I_3\beta K_3}\frac{\sin u}{NR^2} - \frac{c}{I_3K_3}\frac{1}{NR^2}. \tag{16}$$

Finally, from (13), (15) and (16) we can find $\omega(t)$ and $\gamma(t)$.

5. Connection with other rigid body systems

Veselov and Veselova considered the rotation of a rigid body fixed at a point with the nonholonomic constraint $\langle \omega, \gamma \rangle = 0$. The equations of the motion are

$$I\dot{\omega} = I\omega \times \omega + \gamma \times \frac{\partial V}{\partial \gamma} + \lambda\gamma \qquad \dot{\gamma} = \gamma \times \omega, \ \langle \gamma, \omega \rangle = 0. \tag{17}$$

They showed that the system had the invariant measure with density $M = \sqrt{\langle I^{-1}\gamma, \gamma \rangle}$ and gave several integrable cases [5].

It is very interesting that this method could be used for the Veselov and Veselova problem, as well as for the motion of a rigid body fixed at a point without nonholonomic constraint:

$$I\dot{\omega} = I\omega \times \omega + \gamma \times \frac{\partial V}{\partial \gamma} \qquad \dot{\gamma} = \gamma \times \omega. \tag{18}$$

It can be proved.

Theorem 4. Let there be $I_1 \neq I_2 \neq I_3$. The Euler–Poisson equations (18) and the Veselov–Veselova problem (17) are integrable for potentials (12).

Another construction of the integrable perturbations of the motion of a rigid body fixed at a point (18) is given in [15]. Bogoyavlenski proved the integrabilty for arbitrary quadratic potential. He also connected this system, in the case of fixed values of the integral $\langle I\omega, \gamma \rangle = 0$, with integrable systems on the ellipsoid. By the use of the Hamilton–Jacobi method he found, for $\langle I\omega, \gamma \rangle = 0$, the following integrable potentials:

$$V(\gamma) = \sigma_2 \left(c_1 + \sum_{N \geqslant 1} \sum_{k=0}^{N} (-1)^k c_{N+k+1} \binom{N}{k} \sigma_1^{N-k} \sigma_2^k \right) \tag{19}$$

where $\sigma_1 = \frac{\gamma_1^2}{I_1} + \frac{\gamma_2^2}{I_2} + \frac{\gamma_3^2}{I_3} - (I_1^{-1} + I_2^{-1} + I_3^{-1})$, $\sigma_2 = (I_1 I_2 I_3)^{-1}(I_1\gamma_1^2 + I_2\gamma_2^2 + I_3\gamma_3^2)$. Connections between our class of solutions and Bogoyavlenski's class is given by what follows.

Theorem 5. Intersection of the families (19) and (12) consists of the Klebsh–Tisserand potential $V = c(I_1\gamma_1^2 + I_2\gamma_2^2 + I_3\gamma_3^2)$.

Proof. From (9), using algebraic calculations, we can obtain the equation

$$(I_1 - I_2)\gamma_1\gamma_2 \frac{\partial V}{\partial \gamma_3} + (I_3 - I_1)\gamma_3\gamma_1 \frac{\partial V}{\partial \gamma_2} + (I_2 - I_3)\gamma_2\gamma_3 \frac{\partial V}{\partial \gamma_1} = 0. \tag{20}$$

Since (20) is a consequence of (9), all potentials (12) are solutions of (20). On the other hand, from family (19), only the Klebsh–Tisserand potential ($c_1 \neq 0$, $c_i = 0$, $i > 1$) satisfies equation (20). □

6. Summary

Finally, we list the new integrable systems we have obtained in this paper.

The family of integrable potential perturbations parametrized by a pair of two arbitrary functions of the Suslov problem have been constructed. Well known systems of Kharlamova-Zabelina and Kozlov are special cases. Starting from the Suslov, Kharlamova-Zabelina and Kozlov cases, we have obtained new integrable systems by adding gyroscopic force.

The integrability of the Chaplygin problem in the case $I_1 \neq I_2 = I_3$ perturbed with $V = V(\gamma_1, \gamma_2^2 + \gamma_3^2)$ was shown. In this case the additional fourth-degree integral does not follow from the symmetry.

In the case of the dynamical asymmetry $I_1 \neq I_2 \neq I_3$, an infinite-dimensional family of polynomial perturbations of the Chaplygin problem were derived. The same class of potentials serves as integrable perturbations of the classical Euler's case of the rotations of a rigid body fixed at a point, as well for the Veselov–Veselova problem. Until now, the Klebch–Tisserand potential and Bogoyavlenski's quadratic potential for the Euler's case, were the only known integrable potentials of these well known problems.

References

[1] Arnol'd V I, Kozlov V V and Neishtadt A I 1987 *Dynamical Systems III* (Berlin: Springer)
[2] Suslov G K 1946 *Theoretical Mechanics* (Moskow: Gostekhteorizdat) (in Russian)
[3] Chaplygin S A 1976 On rolling of a ball on a horizontal plane *Collected Papers* (Moscow: Nauka) pp 409–28 (in Russian)
[4] Chaplygin S A On rolling of a heavy rotational body on a horizontal plane *Collected Papers* (Moscow: Nauka) pp 363–75 (in Russian)
[5] Veselov A P and Veselova L E 1986 Flows on Lie groups with nonholonomic constraint and integrable nonhamiltonian systems *Funkt. Anal. Prilozh.* **20** 65–6 (in Russian)
[6] Dragovic V 1996 On integrable perturbations of the Jacobi problem for the geodesics on the ellipsoid *J. Phys. A: Math. Gen.* **29** L317–21
[7] Dragovic V 1998 On integrable potentials of billiard within ellipse *Prikl. Mat. Mekh.* **62** (in Russian)
[8] Jovanovic B 1997 Integrable perturbation of billiards on constant curvature surfaces *Phys. Lett.* A **231** 353–8
[9] Dragovic V and Jovanovic B 1997 On integrable potential perturbations of billiard system within ellipsoid *J. Math. Phys.* **38**
[10] Kozlov V V 1995 Some integrable generalizations of the Jacobi problem for the geodesics on the ellipsoid *Prikl. Mat. Mekh.* **59** 3–9 (in Russian)
[11] Fedorov Yu N and Kozlov V V 1995 Various aspects of n-dimensional rigid body dynamics *Am. Math. Soc. Transl. Ser. 2* **168** 141–71
[12] Jovanovic B 1998 Non-holonomic geodesic flows on Lie groups and the integrable Suslov problem on $SO(4)$ *J. Phys. A: Math. Gen.* **31** 1415–22
[13] Kozlov V V 1985 About integration theory of nonholonomic mechanics *Usp. Mekh.* **8** 85–106 (in Russian)
[14] Kharlamova-Zabelina E I 1957 Rapid rotation of a rigid body about a fixed point under the presence of a nonholonomic constraint *Vestnik Moskov Ser. Mat. Mekh.* **12** 25–34 (in Russian)
[15] Bogoyavlenski O I 1985 Integrable cases of dynamics of a rigid body and integrable systems on spheres S^n *Izv. Akad. Nauk SSSR Ser. Math.* **49** 899–915 (in Russian)